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Procedia Engineering 23 (2011) 392 – 396

**Procedia
Engineering**www.elsevier.com/locate/procedia

Power Electronics and Engineering Application

Portfolio selection based on nonparametric estimation and quadric utility maximization framework

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Abstract

This paper adopts the methodology of nonparametric estimation and utility maximization model to explore a portfolio selection problem under the assumption that investors have quadric utility function. First, we obtain the estimated calculation formula for the expected utility by using the nonparametric estimation of portfolio return's density function. Then, the optimal investment strategy for the utility maximization model is obtained. Finally a numerical example based on real data of Chinese stock market is given to show the usefulness and effectiveness of the results.

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Keywords: Portfolio selection; quadric utility function; utility maximization model; nonparametric estimation

1. Introduction

[1] established the theoretical basis for using expected utility function to study uncertain decision problem. [2] introduced the utility maximization method to investigate a portfolio selection problem, and consider the problem of risk aversion and tolerance. [3] studied asset allocation problem for static case by using utility maximization model, and obtained a necessary and sufficient condition for two Fund Separation Theorem to be established. Using utility maximization model, [4] investigated some results on comparative statics of optimal strategy under uncertain market parameter. There are also many other papers have studied portfolio selection problems by using the expected utility maximization framework, such as [5,6,7,8,9], and so on. However, most of these researches are under the assumption that return on assets obey some specific probability distribution type, such as normal distribution, or there are only a

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finite number of possible states in the future. Otherwise, without these condition, previous literature only can discuss the related properties for the expected utility maximization model, but can not solve for the model explicitly or numerically.

It is well known that nonparametric estimation method is appealing in several aspects. One of these is that little or no restrictive prior information on function is needed. Another advantage is that it allows a wide range of data dependence (see [10]), which makes it adaptable in the context of capricious financial market. Therefore, the major objective of this paper is to set up a framework, so that, on one hand, it can directly give out the nonparametric estimated calculation formula for the expected utility under considering investment strategy, on the other hand, it can study the utility maximization portfolio selection problem at the same time.

The reminder of this paper is organized as follows. Firstly, we obtain the estimated calculation formula of expected utility by using nonparametric estimation of the portfolio return's density function. Secondly, we obtain closed form expression for the optimal strategy. Finally, a numerical example base on real data of Chinese stock market is presented to show the validity and the practicability of these results.

2. Market and model setting

Suppose there are n assets (among them, there can be one risk-free assets) in the financial market. Let $\vec{\xi} = (\xi_1, \xi_2, \dots, \xi_n)'$ denote the returns of the assets, $W = (w_1, w_2, \dots, w_n)'$ denote the portfolio of the assets. Here A' denotes the transpose of matrix A . Then, return of portfolio is $\xi_p := \sum_i^n w_i \xi_i = W' \vec{\xi}$, and the utility maximization portfolio selection model can be expressed as

$$\max_W E[U(W' \vec{\xi})], \quad s.t. \quad W' \vec{1} = 1, \quad (1)$$

where $\vec{1} = (1, 1, \dots, 1)'$ represents the vector with every component equal to one, $U(\cdot)$ denotes the utility function corresponding to investor's preference. In order to guarantee the optimal solution exist, $U(\cdot)$ often needs to satisfy some mathematical properties, such as concavity and monotonicity. In this paper, we suppose that $U(\cdot)$ is a quadric utility function, that is say $U(\cdot)$ has the form as follow

$$U(x) = x - 0.5bx^2, \quad b > 0, \quad (2)$$

where b measures the degree of risk aversion.

3. Expected utility maximization model base on nonparametric estimation

Since in general, we know very little information about probability or density function of portfolio's return ξ_p or asset's returns vector $\vec{\xi}$. And on the other hand, we know that nonparametric estimation method does not need to make assumption about the distribution types, and little or no restrictive prior information on function is needed. Therefore, in this paper, we will adopt the nonparametric method to estimate the distribution of ξ_p or $\vec{\xi}$, and then obtain the estimated formula of expected utility. And based on this foundation, we investigate the utility maximization portfolio selection problem. Since in general the asset's returns vector $\vec{\xi}$ is a multidimensional random vector, if we adopt nonparametric method to estimate its density function, the convergence rate of nonparametric estimator would be very slow, which sometimes referred to as the "curse of dimensionality" (see [10]). So in this paper, we estimate the density of return of portfolio, which is only of one dimension, overcoming the problem of "curse of dimensionality", and finally obtain the nonparametric estimation for $E[U(W' \vec{\xi})]$.

Now we introduce some preliminary knowledge of nonparametric theory (see [10]). The nonparametric estimation of probability density function (PDF) $p(x)$ of univariate random variable X with sample set $\{X_1, X_2, \dots, X_T\}$ is

$$\hat{p}(x) = T^{-1}h^{-1} \sum_{i=1}^T k\left(\frac{X_i - x}{h}\right), \quad (3)$$

where $k(\bullet)$ is a kernel function, which, for example, can be chosen as $k(v) = (\sqrt{2\pi})^{-1} e^{-0.5v^2}$.

$G(v) = \int_{-\infty}^v k(t)dt$, and $h = h(T)$ is a smoothing parameter (or alternatively, bandwidth or window width).

It can be proved that the kernel estimator $\hat{p}(x)$ defined in (3) is a consistent estimator of $p(x)$ when kernel function $k(\bullet)$ and bandwidth $h(\bullet)$ satisfy the following condition

- i) $k(\bullet)$ is nonnegative and bounded, $\int k(v)dv = 1$, $k(-v) = k(v)$, $\int v^2 k(v)dv = \kappa_2 > 0$;
- ii) $h(T) \rightarrow 0$ and $Th(T) \rightarrow \infty$ as $T \rightarrow \infty$.

So throughout this paper we always assume that $k(\bullet)$ and $h(\bullet)$ satisfy the conditions (i) and (ii).

In both theoretical and practical settings, the nonparametric kernel estimation is insensitive to the choice of kernel function, but the choice of bandwidth $h(\bullet)$ is a crucial problem. There are many methods for selecting $h(\bullet)$, here we only introduce the common used methods.

Rule-of-thumb: $h \approx 1.06\sigma T^{-0.2}$, where σ is the deviation of X . σ can be estimated by the samples:

$$\hat{\sigma} = \sqrt{(T-1)^{-1} \sum_{i=1}^T (X_i - \bar{X})^2}, \text{ where } \bar{X} = T^{-1} \sum_{i=1}^T X_i.$$

Given investment strategy W and the sample set $\{R_1, R_2, \dots, R_T\}$ of $\bar{\xi}$, then return of portfolio $\xi_p = W'\bar{\xi}$ has the sample set $\{W'R_1, W'R_2, \dots, W'R_T\}$. Adopting the method of nonparametric estimation, nonparametric PDF estimation of ξ_p is

$$\hat{p}(x) = T^{-1}h^{-1} \sum_{i=1}^T k\left(\frac{W'R_i - x}{h}\right), \quad (5)$$

After having estimated the PDF of ξ_p , one can use the plugging-in method to estimate $E[U(W'\bar{\xi})]$ as

$$\hat{E}[U(W'\bar{\xi})] = \int_{-\infty}^{+\infty} U(x) \hat{p}(x) dx = T^{-1}h^{-1} \sum_{i=1}^T \int_{-\infty}^{+\infty} (x - 0.5bx^2) k\left(\frac{W'R_i - x}{h}\right) dx.$$

Let $z = \frac{W'R_i - x}{h}$ in every integral, after integral transformation, and notice that

$$U(x) = x - 0.5bx^2, \int k(z)dz = 1, \int zk(z)dz = 0, \int z^2 k(z)dz = \kappa_2,$$

we can simplify $\hat{E}[U(W'\bar{\xi})]$ as

$$\begin{aligned} \hat{E}[U(W'\bar{\xi})] &= T^{-1} \sum_{i=1}^T \int_{-\infty}^{+\infty} [(W'R_i - zh) - 0.5b(W'R_i - zh)^2] k(z) dz \\ &= T^{-1} \sum_{i=1}^T \int_{-\infty}^{+\infty} [W'R_i - 0.5b(W'R_i)^2 + (bW'R_i - 1)hz - 0.5bh^2 z^2] k(z) dz \\ &= T^{-1} \sum_{i=1}^T [W'R_i - 0.5bW'R_i R_i' W] - 0.5bh^2 \kappa_2 = W'\bar{R} - 0.5bW'\Xi W - 0.5bh^2 \kappa_2, \end{aligned} \quad (6)$$

where $\bar{R} = T^{-1} \sum_{i=1}^T R_i$, $\Xi = T^{-1} \sum_{i=1}^T R_i R_i'$.

We adopt Rule-of-thumb to select bandwidth h , namely $h = 1.06\sigma T^{-0.2}$, where σ is the standard deviation of ξ_p , with its estimator being $\hat{\sigma} = \sqrt{(T-1)^{-1} \sum_{i=1}^T (W'R_i - W'\bar{R})^2}$. Let

$$K = (W'R_1, W'R_2, \dots, W'R_T)' = (R_1, R_2, \dots, R_T)'W = \Re W, M_0 = I - \frac{1}{T} \bar{1} \cdot \bar{1}'.$$

Then $\hat{\sigma}$ can be expressed as $\hat{\sigma} = \sqrt{(T-1)^{-1} K'M_0 K} = \sqrt{(T-1)^{-1} W'\Re'M_0 \Re W} = \sqrt{W'\Omega W}$, where $\Re = (R_1, R_2, \dots, R_T)'$, I denotes the identity matrix of order T , $\Omega = (T-1)^{-1} \Re'M_0 \Re$. Thereby the selected bandwidth can be written as $h = 1.06T^{-0.2} \sqrt{W'\Omega W}$. So, we have

$$\hat{E}[U(W'\tilde{\xi})] = W'\bar{R} - 0.5bW'\Xi W - 0.5b \times 1.06^2 T^{-0.4} W'\Omega W \kappa_2 = W'\bar{R} - 0.5bW'\Sigma W, \quad (7)$$

where $\Sigma = \Xi + 1.06^2 T^{-0.4} \Omega \kappa_2$. Therefore, utility maximization model based on nonparametric estimation method can be written as the following optimization problem

$$\max_W \hat{E}[U(W'\tilde{\xi})] = W'\bar{R} - 0.5bW'\Sigma W, \quad s.t. \quad W'\bar{1} = 1. \quad (8)$$

4. Solving the model

In the following, we study the solving of optimization problem (8). First, we set up the corresponding Lagrange function as $L = W'\bar{R} - 0.5bW'\Sigma W + \lambda(W'\bar{1} - 1)$. The first-order condition is as follow

$$\frac{\partial L}{\partial W} = \bar{R} - b\Sigma W + \lambda \bar{1} = 0, \quad \frac{\partial L}{\partial \lambda} = W'\bar{1} - 1 = \bar{1}'W - 1 = 0. \quad (9)$$

From the first equation of (9), it follow that $W^* = b^{-1}(\Sigma^{-1}\bar{R} + \lambda\Sigma^{-1}\bar{1})$. Substituting it into the second equation of (9) gives $\lambda = (\bar{1}'\Sigma^{-1}\bar{1})^{-1}(b - \bar{1}'\Sigma^{-1}\bar{R})$. Thereby, the optimal solution to optimization (7) is

$$W^* = b^{-1}\Sigma^{-1}\bar{R} + C^{-1}b^{-1}(b - A)\Sigma^{-1}\bar{1}, \quad (10)$$

where $C = \bar{1}'\Sigma^{-1}\bar{1}$, $A = \bar{1}'\Sigma^{-1}\bar{R}$. Substituting (10) into (8), the maximum expected utility is gotten as

$$\hat{E}[U(\tilde{\xi}'W^*)] = 0.5C^{-1}b^{-1}(-b^2 + 2Ab + D), \quad (11)$$

where $B = \bar{R}'\Sigma^{-1}\bar{R}$, $D = BC - A^2 > 0$.

5. Example analysis

We randomly select six stocks from Shenzhen and Shanghai stock exchange. These stock codes are 200053, 600583, 600547, 601699, 002207 and 600111. Selecting the historical daily data of these stocks from June 2, 2008, to May 26, 2011, we get $T = 727$ Day returns samples $\{R_1, R_2, \dots, R_{727}\}$, where the unit of return is $1/50$. We take the kernel function as Gauss kernel function $k(v) = (\sqrt{2\pi})^{-1} e^{-0.5v^2}$, then it

has $\kappa_2 = \int_{-\infty}^{+\infty} v^2 k(v) dv = 1$. Substituting data and through some calculation, we obtain

$$\Sigma = \Xi + 1.06^2 T^{-\frac{2}{5}} \Omega \kappa_2 = \begin{pmatrix} 1.5044 & 0.8313 & 0.6573 & 1.1338 & 1.0767 & 1.2219 \\ 0.8313 & 2.1286 & 0.8773 & 1.5777 & 1.2399 & 1.2671 \\ 0.6573 & 0.8773 & 3.4347 & 1.6536 & 1.2978 & 1.6106 \\ 1.1338 & 1.5777 & 1.6536 & 4.1558 & 1.8767 & 2.2499 \\ 1.0767 & 1.2399 & 1.2978 & 1.8767 & 3.6589 & 1.8154 \\ 1.2219 & 1.2671 & 1.6106 & 2.2499 & 1.8154 & 4.4241 \end{pmatrix}.$$

Let $b = 0.1$, from formula (10)-(11), the maximum utility is $\hat{E}[U(\bar{\xi}'W^*)] = 0.0053$, and the optimal portfolio is

$$W^* = (0.4627, -0.2219, 0.4158, 0.0189, -0.0463, 0.3708)'.$$

Similarly, when $b = 1$, the maximum utility is $\hat{E}[U(\bar{\xi}'W^*)] = -0.5672$, and the optimal portfolio is

$$W^* = (0.5721, 0.2380, 0.2092, -0.0509, 0.0210, 0.0107)'.$$

When $b = 10$, the maximum utility is $\hat{E}[U(\bar{\xi}'W^*)] = -5.8366$, and the optimal portfolio is

$$W^* = (0.5831, 0.2840, 0.1885, -0.0579, 0.0277, -0.0254)'.$$

Acknowledgements

This research is supported by the Humanity and Social Science Foundation of Ministry of Education of China (No. 10YJC790339), Natural Science Foundation of Guangdong Province (No. S2011010005503), and Philosophy and Social Science Foundation of Guangdong Province (No. 09O-19).

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